

# Whispering gallery like modes along pinned vortices

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Employing Unruh's analogy to gravity, we study sound propagation in stationary and locally irrotational vortex flows where the circulation is wound around a long (rotating) cylinder. Apart from the usual scattering solutions, we find anomalous modes which are bound to the vicinity of the cylinder and propagate along its axis – similar to whispering gallery modes. For supersonic flow velocities (corresponding to an effective ergoregion), these modes can even have zero frequency. Thus they should be relevant for the question of stability or instability of this set-up.

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*Introduction* The full characterization of sound modes propagating within a given flow profile is a major problem in fluid dynamics and often reveals very rich physics. Even for stationary flows, which admit a separation ansatz where the linear perturbations can be labeled by their conserved frequency  $\omega$ , this problem is highly non-trivial: For *static* systems, depending on what sort of scenario is considered, the dynamics of perturbations is governed by equations of Schrödinger  $i\partial_t\psi = \mathcal{H}\psi$  or d'Alembert  $\partial_t^2\Phi = \mathcal{D}\Phi$  type. In such cases the full characterization of solutions follows from the spectral analysis of the differential operators  $\mathcal{H}$  or  $\mathcal{D}$ . For *stationary* systems, however, the following inherent difficulty appears. The equations assume the form  $\partial_t^2\Phi + \mathcal{A}\partial_t\Phi = \mathcal{B}\Phi$ , where the two operators  $\mathcal{A}$  and  $\mathcal{B}$  do not commute in general, and therefore their spectral content has no direct significance for the problem at hand. As a result, questions like the completeness of solutions or the existence of unstable modes with  $\Im(\omega) < 0$  are far more difficult to address.

The wave equation for sound in a locally irrotational and stationary background flow has the form mentioned above,  $\partial_t^2\Phi + \mathcal{A}\partial_t\Phi = \mathcal{B}\Phi$ . Precisely the same structure arises for scalar fields propagating in a stationary space-time. Moreover, as discovered by Unruh [1], there is an exact analogy between the two cases: Let us consider a fluid with density  $\varrho$  and velocity  $\mathbf{v}$ , whose pressure  $p$  is a function of  $\varrho$  only  $p = p(\varrho)$ , i.e., the fluid is barotropic. The perturbations, i.e., sound waves, can be parametrized by a single potential  $\Phi$  via  $\delta\mathbf{v} = \nabla\Phi$  and  $\delta\varrho = \varrho\dot{\Phi}/c_s^2$ . Neglecting viscosity, they obey the same wave equation as a scalar field in a curved space-time described by the effective acoustic metric [1]

$$ds^2 = \frac{\varrho}{c_s} (c_s^2 dT^2 - [d\mathbf{R} - \mathbf{v} dT]^2), \quad (1)$$

where  $T$ ,  $\mathbf{R}$  are the laboratory coordinates and  $c_s$  is the speed of sound  $c_s^2 = dp/d\varrho$ . This analogy allows us make use of many geometrical tools and concepts of general relativity in fluid dynamics.

*Vortex flow* In the following, we consider a stationary and locally irrotational  $\nabla \times \mathbf{v} = 0$  flow around a long cylinder. Aligning the coordinate  $Z$ -axis with the symmetry axis of the cylinder, we assume the flow velocity

to be  $\mathbf{v} = v(R)\mathbf{e}_\varphi$  in cylindrical coordinates  $Z, R, \varphi$ . The condition  $\nabla \times \mathbf{v} = 0$  then implies  $\mathbf{v} = \mathbf{e}_\varphi \kappa/R$ , where  $\kappa$  determines the circulation.

For normal fluids, such a profile approximates the stationary flow around a rotating cylinder. Indeed, if the fluid is incompressible  $\nabla \cdot \mathbf{v} = 0$ , the above velocity profile provides an exact solution of the Navier-Stokes equations (similar to a tornado away from the core). While this solution gets modified in more realistic models of normal fluids, for superfluids (such as  $^4\text{He}$  II at low temperatures), vorticity can only occur in the form of vortices with a quantized circulation, which is thus topologically stabilized and not so easy to create (or destroy). The flow profile  $\mathbf{v} = \mathbf{e}_\varphi \kappa/R$  then corresponds to a vortex which is pinned around a long cylinder (e.g., wire), where  $\kappa$  is an integer multiple of  $\kappa_0 = \hbar/M_{\text{He}}$  [2].

In fluid dynamics, it is often useful to express the problem in terms of dimensionless quantities (such as Rossby number) in order to exploit the scaling symmetries. Here, we do the same and choose the sound velocity at infinity  $c_s(R \uparrow \infty) \equiv c_\infty$  as reference scale. Using the circulation  $\kappa$  we get a length  $\mathfrak{L} = \kappa/c_\infty$  and a time scale  $\mathfrak{T} = \kappa/c_\infty^2$ . By rescaling the laboratory coordinates  $T$  and  $\mathbf{R}$ , to get dimensionless  $t$  and  $\mathbf{r}$ , we also find that (1) becomes

$$ds^2 = \frac{\rho}{c} \left[ \left( c^2 - \frac{1}{r^2} \right) dt^2 + 2dt d\varphi - dr^2 - r^2 d\varphi^2 - dz^2 \right] \quad (2)$$

where  $c(r) = c_s(r)/c_\infty$  is the dimensionless sound speed which approaches unity at infinity. Similarly, the normalized density reads  $\rho(r) = \varrho(r)/\varrho_\infty$ . Typically, both decrease a bit when approaching the cylinder where the velocity  $\mathbf{v}$  increases and thus the pressure  $p$  drops.

The coordinates  $t, z$  and  $\varphi$  have their standard ranges, but  $r$  is restricted to  $r \in (r_w, +\infty)$ , where  $r_w$  is the rescaled wire radius. Note that the acoustic metric (2) possesses an *ergoregion*  $g_{00} < 0$  for small enough radii  $r < 1/c$ , if such are in the allowed range, i.e., if  $r_w < 1/c(r_w)$ . Appearance of ergoregion means that the flow velocity  $\mathbf{v}$  exceeds the local speed of sound  $c_s$  somewhere (e.g., near the wire). As we shall see, the presence an ergoregion has profound consequences for the sound modes.

*Geometric acoustics* According to Unruh's analogy to gravity [1], sound modes in the vortex profile are solutions of the wave equation of a massless scalar field in the metric (2). However, before investigating the full wave equation, let us get some insight via the WKB approximation – which amounts to studying sound rays. They are null geodesics in the space-time (2) which can be found in complete analogy to the general relativistic Kepler problem. We search for sound rays  $x^a(\tau)$ , and find that the problem is reduced to quadratures due to existence of four independent first integrals. The space-time (2) admits three symmetries with the Killing vectors  $\partial_t$ ,  $\partial_\varphi$ , and  $\partial_z$ . Via the Noether theorem, this implies the conservation of the energy  $E = \Omega(f\dot{t} + \dot{\varphi})$ , the angular momentum  $J = \Omega(r^2\dot{\varphi} - \dot{t})$ , and the axial momentum  $P = \Omega\dot{z}$ , where the dot denotes  $d/d\tau$ . Here the notation  $\Omega = \rho(r)/c(r)$  and  $f = c^2 - 1/r^2$  is introduced for brevity. Together with the null ray condition  $\dot{x}_a\dot{x}^a = 0$ , we can express all velocities in terms of these first integrals, e.g.,  $\dot{t} = (E - J/r^2)/(\Omega c^2)$  and  $\dot{\varphi} = (E + fJ)/(\Omega c^2 r^2)$ . The remaining radial equation reads

$$\Omega^2 \dot{r}^2 + P^2 = \frac{E^2}{c^2} - \frac{J^2 c^2 + 2EJ}{c^2 r^2} + \frac{J^2}{c^2 r^4} = E^2 - V_{\text{eff}}(r), \quad (3)$$

where we have introduced the effective potential  $V_{\text{eff}}(r)$  which also contains the term  $E^2(1 - 1/c^2)$ .

The sound rays can now be classified by the following arguments. For  $r \uparrow \infty$ , the effective potential  $V_{\text{eff}}(r)$  vanishes. Hence all scattering solutions must have  $E^2 \geq P^2$ . For  $r \downarrow 0$ , on the other hand, the effective potential  $V_{\text{eff}}(r)$  diverges  $V_{\text{eff}}(r \downarrow 0) \downarrow -\infty$  for  $J \neq 0$ . Thus rays are strongly attracted by the vortex in its vicinity. There is a cut-off in  $r$ , however, provided by  $r_w$  (wire radius) where the sound rays are reflected. If, due to  $r_w$ , the potential  $V_{\text{eff}}(r)$  is monotonically *decreasing* for all  $r > r_w$ , then only scattering solutions ( $E^2 \geq P^2$ ) exist. If, however, there are local minima of  $V_{\text{eff}}(r)$  at finite  $r \in [r_w, \infty)$ , bound rays oscillating around them will exist. As one may easily infer from the structure of Eq. (3), this can always be achieved by tuning the angular momentum  $J$ . Choosing, e.g.,  $J = -E$ , we see that  $V_{\text{eff}}(r)$  is strictly negative (assuming  $c \leq 1$  everywhere; see below), and that there will exist rays bouncing off  $r_w$  indefinitely. Note that these are counter-rotating rays, i.e., propagating against the vortex flow.

In the special case of constant  $c$  and  $\rho$  (i.e.,  $\Omega = 1$ ), these qualitative arguments can be made precise. The maximum of  $V_{\text{eff}}(r)$  is at  $r_* = \sqrt{2J/(J + 2E)}$  where  $V'_{\text{eff}}(r_*) = 0$ . Let  $J, E$  be fixed and  $r_w, P$  adjustable. For  $r_w > r_*$  we only have scattering rays, otherwise there exist also bound rays. Generally, for arbitrarily large wire radii  $r_w$ , one can find values of  $J$  and  $E$  for which  $r_*$  is imaginary, and therefore bound rays exist. For small  $J$ , the radius  $r_*$  goes to zero – i.e., bound states require a minimum angular momentum. Finally, for large  $J$ , the radius  $r_* \rightarrow \sqrt{2}$  which is outside the ergoregion at  $r = 1$ .

Surprisingly, in the special case of constant  $c$  and  $\rho$  (i.e.,  $\Omega = 1$ ) as above, the problem of finding null orbits,  $r(\varphi)$ , is exactly soluble in terms of elliptic functions (see [7]), the reason being not-more-than quartic dependence of  $V_{\text{eff}}(r)$  on  $1/r$ , as is also the case in the general relativistic Kepler problem, for example. We note that the  $J^2/r^4$ -term is crucial for the universal existence of bound states as discussed above. As we shall see later, this remains correct for the full wave equation. However, available treatments [4–6] of the subject of propagation of waves in vortex backgrounds introduce assumptions, which effectively eliminate this term. While this leads to the radial equation of simple Bessel form, it also is the reason why the family of bound states reported upon here is not to be found in the literature [4–6].

*Wave acoustics* In what follows the full wave equation will be considered. To this end, let us first discuss the field expressions for  $E, J$ , and  $P$ . They can be obtained from the pseudo energy-momentum tensor [12]

$$T_{ab}[\Phi] = (\partial_a \Phi)(\partial_b \Phi) - \frac{1}{2} g_{ab} g^{cd} (\partial_c \Phi)(\partial_d \Phi), \quad (4)$$

where  $\Phi$  is the velocity potential  $\delta \mathbf{v} = \nabla \Phi$  of the sound waves  $\delta \mathbf{v}$  and  $g_{ab}$  is the acoustic metric (2). For each Killing vector field  $\xi$ , we get a conserved field quantity  $\Xi = \int dS^a T_{ab} \xi^b$  via integrating over the spatial hypersurface  $dS^a$ . As usual, invariance under time-translations leads to the conserved energy,

$$E[\Phi] = \int d^3r \frac{\rho}{2c^2} \left( \dot{\Phi}^2 + c^2 [\nabla \Phi]^2 - \frac{1}{r^4} [\partial_\varphi \Phi]^2 \right). \quad (5)$$

where Minkowski products are implied in the term  $[\nabla \Phi]^2$ , i.e.,  $[\nabla \Phi]^2 = (\partial_r \Phi)^2 + (\partial_z \Phi)^2 + (\partial_\varphi \Phi)^2/r^2$ . The energy functional is positive definite as long as  $c_s^2 > \mathbf{v}^2$  everywhere, i.e.,  $c^2 > 1/r_w^2$ . This is not anymore the case if the ergoregion belongs to the spacetime. The two remaining conserved functionals following from the Killing vector fields  $\partial_z$  and  $\partial_\varphi$ , are the axial momentum

$$P[\Phi] = \int d^3r \frac{\rho}{2c^2} \left( \partial_t \Phi + \frac{1}{r^2} \partial_\varphi \Phi \right) \partial_z \Phi, \quad (6)$$

and similarly the angular momentum  $J[\Phi]$  with  $\partial_z \Phi \rightarrow \partial_\varphi \Phi$ . Furthermore, due to the  $U(1)$  gauge-invariance of the complexified wave equation, the Klein-Fock-Gordon inner product of two solutions

$$(\Phi_1 | \Phi_2) = \frac{i}{2} \int d^3r \Phi_1^* \overleftrightarrow{\left( \partial_t + \frac{1}{r^2} \partial_\varphi \right)} \Phi_2, \quad (7)$$

with  $\Phi_1^* \overleftrightarrow{\partial_a} \Phi_2 = \Phi_1^* \partial_a \Phi_2 - \Phi_2 \partial_a \Phi_1^*$  is conserved, i.e., independent of the Cauchy surface over which it is taken.

*Separation ansatz* In view of the symmetries of our set-up, we approach the problem of solving the wave equation by considering modes specified by the following ansatz, which reflects the structure of the Killing vectors

$$\Phi(t, r, \varphi, z) = \phi(r) \exp\{-i\omega t + im\varphi + ip_z z\}. \quad (8)$$

For such modes the conserved quantities are related to the inner product via  $E[\Phi] = \omega(\Phi|\Phi)$ ,  $P[\Phi] = p_z(\Phi|\Phi)$  and  $J[\Phi] = m(\Phi|\Phi)$ . Since  $(\Phi|\Phi)$  and  $E[\Phi]$  are always real, solutions with complex frequencies must have  $(\Phi|\Phi) = 0$  and  $E[\Phi] = 0$ . In the absence of an ergoregion,  $E[\Phi]$  is positive definite and all frequencies are real, i.e., the flow is linearly stable [14]. Furthermore, the frequency  $\omega$  and the pseudo-norm  $(\Phi|\Phi)$  of our modes have the same sign in this case. As a result, creation and annihilation operators are associated to modes with positive and negative frequencies, respectively. In the case with an ergoregion, the energy can become negative and hence this property is no longer true. This can lead to interesting and related phenomena such as super-radiance [8] and the Klein paradox [9]. Since a given frequency  $\omega > 0$  can be associated to both, creation and annihilation operators, one can have a mixing of the two and thus phenomena like particle creation.

By inserting the above separation ansatz (8) into the wave equation we reduce it to a single ordinary differential equation (in radial direction):

$$\left[ -\frac{1}{r\rho} \frac{d}{dr} r\rho \frac{d}{dr} + \omega^2 \left[ 1 - \frac{1}{c^2} \right] + \frac{m^2 c^2 + 2m\omega}{c^2 r^2} - \frac{m^2}{c^2 r^4} \right] \phi = \mathcal{H}\phi = (\omega^2 - p_z^2) \phi = \lambda\phi. \quad (9)$$

This main equation of our sound-propagation problem has several interesting features [7]. First of all, because  $\rho \rightarrow 1$  and  $c \rightarrow 1$  at  $r \uparrow \infty$ , the solutions  $\phi(r)$  at large  $r$  are either oscillating, for  $\omega^2 > p_z^2$ , or exponentially decaying, for  $\omega^2 < p_z^2$ . In complete analogy to the ray problem, we call the first type of solutions the *scattering modes*, and the second the *bound-state modes*. Finding these modes is then reduced to an eigenvalue problem  $\mathcal{H}\phi = \lambda\phi$  with  $\mathcal{H} = \mathcal{D} + \mathcal{V}$ , where  $\mathcal{D}$  stands for the “kinetic part” involving  $r$ -derivatives and  $\mathcal{V}$  is the effective potential. Note that  $V_{\text{eff}}(r)$  in the sound-ray problem corresponds to  $\mathcal{V}$  on identification of  $E$  and  $J$  with  $\omega$  and  $m$ , respectively. For the scalar product

$$\{\phi_1|\phi_2\} = \int_{r_w}^{\infty} dr \rho(r) r \phi_1^*(r) \phi_2(r), \quad (10)$$

the operator  $\mathcal{H}$  is self-adjoint if the Neumann boundary condition at  $r_w$  is assumed  $\phi'(r_w) = 0$ , which would just reflect the fact that perturbations can not penetrate the wire. Even though neither is  $\mathcal{H}$  the Hamiltonian of the problem, nor is the scalar product (10) distinguished by the geometry, both of these elements suffice for the following general statements. First of all,  $\mathcal{H}$  has bound states,  $\lambda < 0$ , only if it one can find test functions  $\psi$  such that  $\{\psi|\mathcal{H}|\psi\} < 0$ . The kinetic part,  $\mathcal{D}$ , is always non-negative, and therefore bound states can only exist if  $\mathcal{V}$  is sufficiently negative. A lower bound on the eigenvalue  $\lambda$  can be obtained via the minimum of  $\mathcal{V}(r)$ . Then, for general profiles of  $c(r)$  and  $\rho(r)$  with the aforementioned

asymptotics, we get the following statements:

- 1) Bound states with  $\omega = 0$  can only exist if the velocity becomes supersonic somewhere  $c < 1/r$ , i.e., if the acoustic spacetime has an ergoregion, for otherwise  $\mathcal{V}(\omega = 0)$  is non-negative – consistent with  $E[\Phi] = \omega(\Phi|\Phi)$ .
- 2) Bound states with  $m = 0$  can only exist if  $c_s$  becomes sufficiently smaller than  $c_\infty$  near the wire. The mechanism for bound states in this case is just the total reflection from the region with larger speed of sound – which can also occur in a non-rotating fluid.
- 3) Independently of the mechanism of total reflection, caused by the  $\omega^2$  term in  $\mathcal{V}$ , the other terms in  $\mathcal{V}$  allow for bound states with  $m\omega > 0$  (i.e., co-rotating) only in the presence of an ergoregion, i.e., if the flow becomes supersonic  $v(r) > c_s(r)$  somewhere.
- 4) For any radius  $r_w > 0$  of the wire there are always bound states for some values of  $m$  and  $\omega$ .

These results can be shown in analogy to the sound-ray case. Let us first note that the Bernoulli theorem ( $\mathbf{v}^2/2 + h(\varrho) = \text{const}$  for free stationary flow) implies that the enthalpy  $h(\varrho)$  drops towards the center, because  $\mathbf{v}$  increases. Thus the pressure  $p$  and the density  $\varrho$  must decrease near the wire. Furthermore, for most fluids the Grüneisen parameter  $\propto dc_s/d\varrho$  is positive (e.g., for  $^4\text{He}$  II at  $T = 0$ , we have  $dc_s/d\varrho \in [2.2, 2.9] \times c_s/\varrho$ , [10]), and therefore the speed of sound is also a monotonically decreasing function of  $r$ . Thus the term  $\omega^2(1 - 1/c^2)$  in  $\mathcal{V}$  is negative. For sufficiently large  $\omega$ , one could get bound states for  $m = 0$  via total reflection, see point 2. However, if the change in  $c$  is small and thus the required frequencies are too large, the underlying fluid dynamic description might not be applicable anymore.

In contrast, the bound states for  $m \neq 0$  mentioned in point 4 can occur for smaller values of  $\omega$  and at larger length scales. Let us consider the operator  $\mathcal{H}$ , where in the potential  $\mathcal{V}$  we may vary the parameters  $m$  and  $\omega$ . The remaining freedom of choosing  $p_z$  can be used to adjust  $\lambda$ . For an arbitrary test function  $\psi$ , the kinetic part  $\{\psi|\mathcal{D}|\psi\} > 0$  of the expectation value of  $\mathcal{H}$  is independent of  $m$  and  $\omega$ , but for counter-rotating modes the expectation value of  $\mathcal{V}$  can be made arbitrarily negative. For instance for  $\omega = -m$  the expectation value of  $\mathcal{V}$  is negative for all test functions and scales as  $m^2$ . Thus, if  $m^2$  is large enough, we get  $\{\psi|\mathcal{H}|\psi\} < 0$ , i.e., bound states must exist.

*Case of constant  $\rho$  and  $c$*  The arguments stated above prove the existence of the bound states for large enough  $m$ , but they do not provide information about how large  $m$  must be for a given set-up and how  $\omega$  depends on  $p_z$ , for example. To get this information, we numerically solve Eq. (9) for the special case of constant  $\rho$  and  $c$ . For moderate velocities  $\mathbf{v}$  (sufficiently below the speed of sound), this should be a reasonably good approximation.

In Fig. 1, we plot examples of the dispersion relations  $\omega(p_z)$  of the bound states. In the right plot, we choose  $r_w = 5$ , in which case that the flow velocity does not ex-

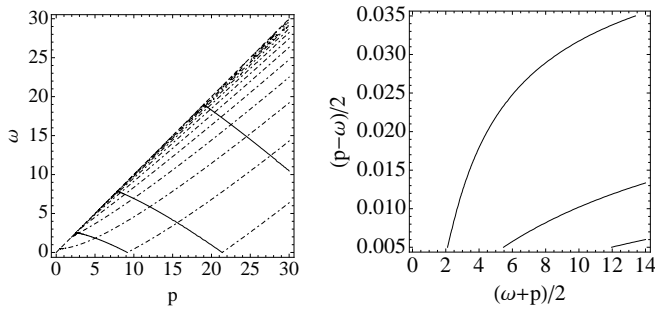


FIG. 1: Dispersion relations for bound states. (Left) with ergoregion,  $r_w = 0.3$ ; (solid) co-rotating waves  $m = 5$ ; (dashed) counter-rotating waves  $m = -5$ . (Right) without ergoregion,  $r_w = 5$ , rotated and zoomed view for counter-rotating waves,  $m = -3$ . The region  $\omega \geq |p|$  is filled with scattering states.

ceed  $c_\infty/5$  and thus the density should be constant up to a few percent. In agreement with statements **2** and **3**, we only find counter-rotating bound states, here we plotted modes with  $m = -3$ . In agreement with statement **1**, there is a gap for such modes, i.e., a minimum frequency, of  $\omega_{\min} \approx 1.7$ . Above this gap, the dispersion relation is almost linear and very close to the  $\omega = p_z$  line – note the different scales on the  $(p - \omega)/2$  and the  $(p + \omega)/2$  axes – which means that these modes propagate with almost the sound speed along the vortex.

For comparison, we plotted an example with an ergoregion  $r_w = 0.3$  in Fig. 1 (left). Even though such a profile is probably hard to realize experimentally (especially with  $\varrho \approx \text{const}$ ), one could imagine keeping the density nearly constant by applying a suitable external potential  $V$  such that the stationary Bernoulli equation reads  $v^2/2 + h(\varrho) + V = \text{const}$ . Furthermore, one could rotate the cylinder with the same velocity as the innermost layer of the fluid in order to stabilize the set-up better. This profile allows co-rotating bound states (here  $m = 5$ ) which have a negative group velocity in addition to the counter-rotating modes (here  $m = -5$ ). In this case, neither co- nor counter-rotating modes have a gap – at discrete values of  $p_z$ , the frequency and thus the energy  $E = \omega(\Phi|\Phi)$  vanishes. Such zero-energy modes are not uncommon in problems involving perturbations of vortices [11] and may indicate an instability.

**Conclusions** For a vortex pinned at a wire, we studied sound propagation via Unruh’s analogy to gravity. On general grounds, we predict the existence of bound states – whispering gallery like modes – based on the geometric acoustics approximation as well as the full wave equation.

It should be possible to verify the characteristics of the bound states in experiments. For example, let us consider superfluid  $^4\text{He}$  with a typical speed of sound  $c_s = 2.4 \times 10^4 \text{cm/s}$ . For a singly quantized vortex we get  $\kappa = 1.6 \times 10^{-4} \text{cm}^2/\text{s}$ . This leads to the length scale  $\mathfrak{L} = 6 \times 10^{-9} \text{cm}$  which is smaller than the van der Waals radius of Helium, and to the frequency scale

$c_\infty^2/\kappa = 3.6 \times 10^{12} \text{Hz}$  above the roton frequency. At these scales, fluid dynamics (which is the basis of our sound description) is not valid anymore. Thus, for observing these bound states, it is probably better to trap many circulation around the wire, which linearly increases  $\kappa$  and thus  $\mathfrak{L}$ , while decreasing the frequency of the modes.

We remark that the family of bound states presented here is distinct from the phenomenon of Kelvin waves, known from normal- and superfluid dynamics. The vortex considered in this Letter is pinned and not allowed to move or be deformed. Accordingly, the dispersion relation (e.g., in Fig. 1) of the modes considered here is much stiffer and more sound-like than that of the soft Kelvin waves with  $\omega(p_z) \sim \kappa p_z^2 \log(p_z)/2$  for small  $p_z$ , see [2]. In fact, in the absence of ergoregion the bound states discussed here do not even exist for small  $p_z$ , where the above Kelvin formula is intended to be applicable.

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- [1] W.G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981).
  - [2] R.J. Donnelly, *Quantized vortices in Helium II* (Cambridge University Press, 1991).
  - [3] E.B. Sonin, Rev. Mod. Phys. **59**, 87 (1987).
  - [4] A.L. Fetter, Phys. Rev. **136**, A1488 (1964).
  - [5] M. Stone, Phys. Rev. B **61**, 11780 (2000).
  - [6] M. Flaig, U.R. Fischer, Phys. Rev. B **74**, 224503 (2006).
  - [7] P. Marecki, arXiv:1110.0115 (2011).
  - [8] M. Richartz et al., Phys. Rev. **D80**, 124016 (2009).
  - [9] S.A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, 1989).
  - [10] J. Brooks and R.J. Donnelly, J. Phys. Chem. Ref. Data **6**, 51 (1977).
  - [11] D. Fabre, et al., J. Fl. Mech. **551**, 235 (2006).
  - [12] See, e.g., M. Stone, Phys. Rev. E **62**, 1341 (2000); M. Stone, *Phonons and forces: Momentum versus pseudomomentum in moving fluids*, in M. Novello, M. Visser, and G. Volovik (editors), *Artificial Black Holes* (World Scientific, Singapore, 2002).
  - [13] U. R. Fischer and M. Visser, Phys. Rev. Lett. **88**, 110201 (2002).
  - [14] This argument can be made more precise. For  $r_w > 1$ , a Hamiltonian formulation of the problem exists [7] with the Hamiltonian  $H$  being a self-adjoint operator acting on a Hilbert space. Thus in this case all  $\omega$  (eigenvalues of  $H$ ) are real, and the family of corresponding modes is complete in the usual sense. For  $r_w < 1$ , on the other hand, the operator  $H$  is only a symmetric operator on a Krein space, and the presence of complex  $\omega$  cannot be excluded *a priori*. However, based on the  $(p_z, m, r_w)$ -dependence of real  $\omega$ , we conjecture that they do not appear [7]. To the best of our knowledge, neither the problem of existence of complex frequencies, nor the issue of completeness of the eigenmodes of  $H$  has been settled in the literature for this case.